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The $\overline{\partial}$-approach to the dispersionless (2+1)-Harry Dym hierarchy

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Abstract

A $\overline{\partial}$-approach is adopted to study the dispersionless Harry Dym (dHD) hierarchy. Moreover, this formulism is applied to construct some explicit solutions of the dHD hierarchy.

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1. Introduction

The Lax formalism of dispersionless integrable systems (see [24] for a review) is defined by algebra Λ of Laurent series $\sum_i a_i(X, T_1, T_2, \ldots)p^i$. One can check that, with respect to the Poisson bracket $\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial X} - \frac{\partial f}{\partial X}\frac{\partial g}{\partial p}$ where f, g are all in the algebra Λ, there are only three closed subalgebra decompositions of Λ

$$\Lambda = \Lambda_{\geqslant k} \oplus \Lambda_{<k}, \qquad k = 0, 1, 2,$$

with $\Lambda_{\geqslant k} = \left\{\sum_{\geqslant k} a_i(X, T_1, T_2, \ldots)p^i\right\}$, and $\Lambda_{<k} = \left\{\sum_{<k} a_i(X, T_1, T_2, \ldots)p^i\right\}$. Hence from this classification, one can introduce dispersionless Lax hierarchies as follows,

$$\frac{\partial \mathcal{L}}{\partial T_n} = \{B_n, \mathcal{L}\}, \qquad B_n = (\mathcal{L}^n)_{\geqslant k}$$

in which the dispersionless Kadomtsev–Petviashvili (dKP) [11, 13, 19], the dispersionless modified Kadomtsev–Petviashvili (dmKP) [20, 21, 4] and the dispersionless Harry Dym (dHD) [21] hierarchies are three basic Lax hierarchies as they correspond to cases $k = 0, 1$ and 2, respectively. Besides, the Lax formalism of the dispersionless Toda (dToda) hierarchy was established by considering a pair of Lax operators [12, 24].

Different methods have been used to study dispersionless equations and hierarchies. In particular, for those finite-dimensional reductions of dispersionless Lax hierarchies that are hierarchy flows of hydrodynamic type and can be diagonalized by Riemann invariants, one can solve them by the hodograph method so that their solutions are expressed in implicit form (see, e.g., [11, 13, 12, 8]). Recently, the quasiclassical $\bar{\partial}$-dressing approach to the dKP, dmKP and dToda hierarchies has been proposed and studied well by Konopelchenko *et al* [16, 17]. This approach can be viewed as the quasiclassical limit of the nonlocal Riemann–Hilbert problem (see, e.g., [15]) and directly relates dispersionless integral hierarchies with quasiconformal mapping on the plane. Hence one can present some explicit solutions of dispersionless integral hierarchies by the widely studied theory of quasiconformal mappings. In [18], Konopelchenko *et al* justified that some solutions may not be obtained by standard hodograph transformation methods. Therefore, motivated by this observation, we shall show the $\bar{\partial}$-approach may complement the hodograph method [6, 7] for constructing solutions of the dHD system.

In this paper, we shall adapt Konopelchenko's formalism to study the dHD hierarchy. In this approach the analytic property of the S function (or the WKB phase function of the Baker–Akhiezer function [11]) defined by the Beltrami equation plays an essential role for deriving an explicit solution of the dHD hierarchy. Our result presents a new class of finite-dimensional solutions to the dHD hierarchy which were not obtained in the previous literature [6–8]. This paper is organized as follows. We review the $\bar{\partial}$-problem of the HD hierarchy and introduce the reciprocal formula between the HD and mKP hierarchies in section 2. In section 3, we introduce the quasiclassical $\bar{\partial}$-problem of the dHD hierarchy. In section 4, we derive the reciprocal formula between the dHD and dmKP hierarchies from a new approach, the $\bar{\partial}$-approach. In sections 5 and 6, we use the quasiclassical $\bar{\partial}$-problem to derive the dHD hierarchy and to provide explicit solutions for the dHD flows.

2. The $\overline{\partial}$-problem of the HD hierarchy

The HD hierarchy is an infinite set of the compatibility conditions for the system

$$L\Psi = \frac{1}{\lambda}\Psi,$$
$$\frac{\partial \Psi}{\partial t_n} = (L^n)_{\geqslant 2}\Psi, \qquad n \geqslant 2,$$

where $L = u_1(\vec{t})\partial + u_0(\vec{t}) + u_{-1}(\vec{t})\partial^{-1} + u_{-2}(\vec{t})\partial^{-2} + \cdots$, $\partial = \partial/\partial x$, $\vec{t} = (x, t_2, t_3, \ldots)$, λ is a spectral parameter, $(L^n)_{\geqslant 2}$ denotes the pure differential part of the operator L^n with terms of degree larger than 2, and $\Psi(\vec{t}, \lambda)$ is a wavefunction of the HD hierarchy. Since $(L)_{\geqslant 2} = 0$, u_i do not depend on t_1. The first nontrivial equation of Lax flow is the HD equation in 2+1 dimensions [14],

$$4u_{1t} = u_1^3 u_{1xxx} - \frac{3}{u_1}\left(u_1^2 \int^x \left(\frac{1}{u_1}\right)_y \mathrm{d}x\right)_y \tag{1}$$

which is the equation for coefficient u_1 as a function of the first three independent variables $x, t_2 = y, t_3 = t$. In particular, it is the compatibility of the Lax pair

$$M_2\Psi = \left(\frac{\partial}{\partial y} - u_1^2\frac{\partial^2}{\partial x^2}\right)\Psi = 0, \tag{2}$$

$$M_3\Psi = \left(\frac{\partial}{\partial t} - u_1^3\frac{\partial^3}{\partial x^3} - \frac{3}{2}u_1^2\left[\frac{\partial u_1}{\partial x} - \left(\frac{\partial}{\partial x}\right)^{-1}\left(\frac{1}{u_1}\right)_y\right]\frac{\partial^2}{\partial x^2}\right)\Psi = 0. \tag{3}$$

The $\overline{\partial}$-*problem* of the HD hierarchy is the study of the Cauchy integral formula of the wavefunction Ψ of the Lax pair (2), (3) [10] and their higher order operators $M_4, M_5, \ldots$ [15]. The $\overline{\partial}$-dressing formula, i.e. the Cauchy integral formula, which we consider is written as

$$\chi(\vec{t}, \lambda, \bar{\lambda}) = 1 + \frac{1}{2\pi\mathrm{i}} \iiiint \frac{\chi(\vec{t}, \mu, \bar{\mu}) R(\vec{t}, \mu, \bar{\mu}, \xi, \bar{\xi})}{\xi - \lambda} \, \mathrm{d}v_\mu \, \mathrm{d}v_\xi, \tag{4}$$

with

$$\Psi(\vec{t}, \lambda, \bar{\lambda}) = \chi(\vec{t}, \lambda, \bar{\lambda}) \exp\left(\frac{f(\vec{t})}{\lambda} + \frac{t_2}{\lambda^2} + \frac{t_3}{\lambda^3} + \cdots\right), \tag{5}$$

$$R(\vec{t}, \mu, \bar{\mu}, \xi, \bar{\xi}) = R_0(\mu, \bar{\mu}, \xi, \bar{\xi}) \exp\left(f(\vec{t})\left(\frac{1}{\mu} - \frac{1}{\xi}\right) + t_2\left(\frac{1}{\mu^2} - \frac{1}{\xi^2}\right) + t_3\left(\frac{1}{\mu^3} - \frac{1}{\xi^3}\right) + \cdots\right),$$

$$\mathrm{d}v_\mu = \mathrm{d}\mu \wedge \mathrm{d}\bar{\mu}, \qquad \mathrm{d}v_\xi = \mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}.$$

The function $f(\vec{t})$ has to fulfil systems of nonlinear constraints. For the HD case, using (2), (3) and (5), f satisfies

$$u_1 = \frac{1}{f_x}, \tag{6}$$

$$f_y - \frac{f_{xx}}{f_x^2} - \frac{2}{f_x}\frac{\chi_{0x}}{\chi_0} = 0, \tag{7}$$

$$f_t - \frac{f_{xxx}}{f_x^3} + \frac{3}{2} f_y^2 + \frac{3}{2}\frac{f_{xx}^2}{f_x^4} - \frac{3}{2}\frac{\chi_{0y}}{\chi_0} - \frac{3}{2f_x^2}\frac{\chi_{0xx}}{\chi_0} = 0, \tag{8}$$

with $\chi_0 = \chi(\lambda = 0)$, $\chi_{0x} = \frac{\partial\chi}{\partial x}(\lambda = 0)$, $\chi_{0y} = \frac{\partial\chi}{\partial y}(\lambda = 0)$, etc. The nonlinear constraints (6)–(8) can also be understood through the famous reciprocal formula between the mKP and HD hierarchies [10, 23, 9]:

$$x' = f(\vec{t}), \qquad t_n' = t_n, \qquad n \geqslant 2, \tag{9}$$

$$x = \phi'(\vec{t}'), \qquad t_n = t_n', \qquad n \geqslant 2, \tag{10}$$

$$\chi(\vec{t}, \lambda, \bar{\lambda}) = \chi'(x', t_2', \ldots, \lambda, \bar{\lambda}), \tag{11}$$

$$\Psi(\vec{t}, \lambda, \bar{\lambda}) = \Psi'(x', t_2', \ldots, \lambda, \bar{\lambda}), \tag{12}$$

where $\vec{t}_n' = (x' = t_1', \ldots)$ are time variables of the mKP hierarchy, f is defined as in (5) and χ', Ψ' are the normalized wavefunction and the wavefunction of the corresponding mKP hierarchy. Note that the reciprocal transformation is isospectral. Under the reciprocal formula (9)–(11), formula (7) and (8) are reduced to:

$$M_2'\phi' = \left(\frac{\partial}{\partial y'} - \frac{\partial^2}{\partial x'^2} - 2v_0\frac{\partial}{\partial x'}\right)\phi' = 0,$$

$$M_3'\phi' = \left(\frac{\partial}{\partial t'} - \frac{\partial^3}{\partial x'^3} - 3v_0\frac{\partial^2}{\partial x'^2} - \frac{3}{2}\left[v_0^2 + \frac{\partial v_0}{\partial x'} + \left(\frac{\partial}{\partial x'}\right)^{-1}(v_0)_y\right]\frac{\partial}{\partial x'}\right)\phi' = 0,$$

with $v_0 = -\frac{\chi_{0x'}'}{\chi_0'} = -\frac{\chi_{0x'}}{\chi_0}$, which is just the Lax pair of the mKP equation:

$$v_{0t'} = -\tfrac{3}{2}v_0^2 v_{0x'} + \tfrac{3}{2}v_{0x'}\partial_{x'}^{-1}v_{0y'} + \tfrac{3}{4}\partial_{x'}^{-1}v_{0y'y'}.$$

Hence ϕ' is an mKP wavefunction.

3. Quasiclassical $\bar{\partial}$-problem

To derive the quasiclassical limit of the $\bar{\partial}$-problem (4), we introduce the slow time variables $\vec{T} = \epsilon\vec{t}$, $\vec{T} = (X, T_2, T_3, \ldots)$, and let $\epsilon \to 0$. In this limit, we look for solutions χ, f of the form

$$\exp\left(\frac{1}{\lambda} f\left(\frac{\vec{T}}{\epsilon}\right) + \frac{1}{\lambda^2}\frac{T_2}{\epsilon} + \frac{1}{\lambda^3}\frac{T_3}{\epsilon} + \cdots\right) \to \exp\left(\frac{S_0(\vec{T},\lambda)}{\epsilon} + \mathcal{O}(\epsilon)\right) \tag{13}$$

$$S_0(\vec{T},\lambda) = \frac{1}{\lambda}\tilde{f}(\vec{T}) + \frac{T_2}{\lambda^2} + \frac{T_3}{\lambda^3} + \cdots \tag{14}$$

$$\chi(\vec{T}/\epsilon, \lambda, \bar{\lambda}) = \hat{\chi}(\vec{T}, \epsilon, \lambda, \bar{\lambda}) \exp\left(\frac{\tilde{S}(\vec{T},\lambda,\bar{\lambda})}{\epsilon}\right), \tag{15}$$

$$\hat{\chi}(\vec{T}, \epsilon, \lambda, \bar{\lambda}) = \sum_{n=0}^{\infty} \hat{\chi}_n(\vec{T}, \lambda, \bar{\lambda})\epsilon^n, \tag{16}$$

$$\tilde{S}(\vec{T}, \lambda, \bar{\lambda}) \to \mathcal{O}(|\lambda|^{-1}), \qquad \text{as} \quad \lambda \to \infty \tag{17}$$

$$\tilde{S}(\vec{T}, \lambda, \bar{\lambda}) \to \hat{S}_0(\vec{T}) + \hat{S}_1(\vec{T})\lambda + \hat{S}_2(\vec{T})\lambda^2 + \cdots, \qquad \text{as} \quad \lambda \to 0, \tag{18}$$

and consider scattering data R_0 of the form

$$R_0(\mu, \bar{\mu}, \xi, \bar{\xi}) = \sum_{k=0}^{\infty} \Gamma_k(\mu, \bar{\mu})\epsilon^{k-1}\delta_\mu^{(k)}(\mu - \xi),$$

where $\delta_\mu^{(k)}(\mu - \xi) = \partial_\mu^k \delta(\mu - \xi)$. Then the quasiclassical limit of (4) yields

$$\partial_{\bar{\lambda}}\hat{\chi} + \frac{1}{\epsilon}\hat{\chi}\,\partial_{\bar{\lambda}} S = \int_C \hat{\chi} \exp\left(\frac{1}{\epsilon}(S(\mu) - S(\lambda))\right) \sum_{k=0}^{\infty} \Gamma_k(\mu, \bar{\mu})\epsilon^{k-1}\delta_\mu^{(k)}(\mu - \lambda)\,\mathrm{d}v_\mu, \tag{19}$$

where

$$S = S_0 + \tilde{S}. \tag{20}$$

The terms of $1/\epsilon$ in (19) result in the *quasiclassical $\bar{\partial}$-problem*

$$\frac{\partial S}{\partial \bar{\lambda}} = W\left(\lambda, \bar{\lambda}, \frac{\partial S}{\partial \lambda}\right), \qquad W\left(\lambda, \bar{\lambda}, \frac{\partial S}{\partial \lambda}\right) = \sum_{k=0}^{\infty} \Gamma_k(\lambda, \bar{\lambda}) \left(\frac{\partial S}{\partial \lambda}\right)^k. \tag{21}$$

4. The reciprocal formula

We now justify that the quasiclassical limit (13)–(21) under the reciprocal formulae (9)–(12) is indeed transformed into that of the mKP hierarchy. More precisely,

Proposition 1. *Suppose (13)–(18) hold. Define $\vec{T}' = \epsilon\vec{t}'$, $\vec{T}' = (X', T_2', T_3', \ldots)$ and $\vec{t}'$ is the corresponding mKP-time variable defined by (9). Then*

$$X' = \tilde{f}(\vec{T}), \qquad T_n' = T_n, \qquad n \geqslant 2, \tag{22}$$

$$X = \Phi'(\vec{T}'), \qquad T_n = T_n', \qquad n \geqslant 2, \tag{23}$$

and Φ' satisfies:

$$\frac{\partial \Phi'}{\partial T_n'} = \{((\tilde{\mathcal{L}}')^n)_{\geqslant 1}, \Phi'\}_{[0]}, \tag{24}$$

with $\tilde{\mathcal{L}}' = p + \tilde v_0 + \frac{\tilde v_{-1}}{p} + \frac{\tilde v_{-2}}{p^2} + \cdots$, $(f)_{[k]} = a_k$, *for* $f = \sum a_i(\vec T)p^i$. *Moreover, let*

$$S'(T', \lambda, \bar\lambda) = S(\vec T, \lambda, \bar\lambda). \tag{25}$$

Then

$$S' = S'_0 + \tilde S', \tag{26}$$

$$S'_0 = \frac{X'}{\lambda} + \frac{T'_2}{\lambda^2} + \frac{T'_3}{\lambda^3} + \cdots, \tag{27}$$

$$\tilde S'(\vec t', \lambda, \bar\lambda) \to \mathcal{O}(|\lambda|^{-1}), \qquad as \quad \lambda \to \infty, \tag{28}$$

$$\tilde S'(\vec T', \lambda, \bar\lambda) \to \hat S'_0 + \hat S'_1\lambda + \cdots, \qquad as \quad \lambda \to 0, \tag{29}$$

$$\frac{\partial S'}{\partial\bar\lambda} = W\left(\lambda, \bar\lambda, \frac{\partial S'}{\partial\lambda}\right). \tag{30}$$

Proof. For convenience, we quote the proof in [5] to show (22)–(24). Using (13) and (14), we have

$$\exp\left(\frac{1}{\lambda}f\left(\frac{\vec T}{\epsilon}\right) + \frac{1}{\lambda^2}\frac{T_2}{\epsilon} + \frac{1}{\lambda^3}\frac{T_3}{\epsilon} + \cdots\right) \to \exp\left(\frac{1}{\epsilon}\left(\frac{1}{\lambda}\tilde f(\vec T) + \frac{T_2}{\lambda^2} + \frac{T_3}{\lambda^3} + \cdots\right) + \mathcal{O}(\epsilon)\right). \tag{31}$$

Besides, (9), $\vec T = \epsilon\vec t$, and the assumption $\vec T' = \epsilon\vec t'$ imply

$$\exp\left(\frac{1}{\lambda}f\left(\frac{\vec T}{\epsilon}\right) + \frac{1}{\lambda^2}\frac{T_2}{\epsilon} + \frac{1}{\lambda^3}\frac{T_3}{\epsilon} + \cdots\right) = \exp\left(\frac{x'}{\lambda} + \frac{t'_2}{\lambda^2} + \frac{t'_3}{\lambda^3} + \cdots\right)$$
$$= \exp\left(\frac{1}{\epsilon}\left(\frac{X'}{\lambda} + \frac{T'_2}{\lambda^2} + \frac{T'_3}{\lambda^3} + \cdots\right)\right). \tag{32}$$

Comparing (31) and (32), we conclude (22). Hence we define (23) with Φ' being the inverse map of $\tilde f$.

On the other hand, by (10), $T_n = \epsilon t_n$ and $T'_n = \epsilon t'_n$, we obtain $X = \epsilon\phi'\left(\frac{\vec T'}{\epsilon}\right)$. Thus $\phi'\left(\frac{\vec T'}{\epsilon}\right) = \frac{1}{\epsilon}\Phi'(\vec T')$. Plugging this identity and $T'_n = \epsilon t'_n$ in the Lax representation of the mKP hierarchy

$$\frac{\partial\phi'}{\partial t'_n}(\vec t') = (L'^n)_{\geqslant 1}\,\phi'(\vec t'),$$

where $L' = \partial' + v_0 + v_{-1}\partial'^{-1} + v_{-2}\partial'^{-2} + \cdots$, $\partial' = \partial/\partial x'$, $v_i(\vec T'/\epsilon) = \tilde v_i(\vec T') + \mathcal{O}(\epsilon)$. We find, as $\epsilon \to 0$, $\frac{\partial\phi'}{\partial t'_n} = \frac{\partial\Phi'}{\partial T'_n}$ and

$$(L'^n)_{\geqslant 1}\,\phi'(\vec t') = \frac{1}{\epsilon}(L'^n)_{\geqslant 1}\,\Phi' \to \frac{1}{\epsilon}(L'^n)_{[1]}\partial'\Phi' = \{((\tilde{\mathcal{L}}')^n)_{\geqslant 1}, \Phi'\}_{[0]}.$$

So (24) is proved. At last, after identifying these time variables of the mKP and dmKP hierarchies, we can repeat the process in section 3 to justify (26)–(30). □

In [16], the condition (26)–(30) is shown to be equivalent to the existence of a dispersionless mKP hierarchy. Besides, we are going to show that (13)–(18) is equivalent to the existence of a dHD hierarchy in the next section. Therefore, via the $\bar\partial$-approach, we derive a reciprocal formula or a Miura transformation between the dHD and dmKP hierarchies.

5. Derivation of the dHD hierarchy

We are going to derive the dHD hierarchy by the linear Beltrami equation

$$\frac{\partial}{\partial\bar{\lambda}}\left(\frac{\partial S}{\partial T_j}\right) = \dot{W}\left(\lambda, \bar{\lambda}, \frac{\partial S}{\partial\lambda}\right)\frac{\partial}{\partial\lambda}\left(\frac{\partial S}{\partial T_j}\right), \tag{33}$$

which is obtained by taking derivatives of (21). Note that $\dot{W}$ means the derivative with respect to the variable $\frac{\partial S}{\partial\lambda}$. The Beltrami equation possesses the following properties [2, 1] which will be used in our derivation:

- (Vekua's theorem) Under mild conditions on W, the only solution Ψ of the Beltrami equation (33) such that $\Psi = 0$ at ∞ is $\Psi \equiv 0$.
- (Ring of symmetry) If $\Psi_1, \ldots, \Psi_N$ are solutions of the Beltrami equation (33), a differentiable function $f(\Psi_1, \ldots, \Psi_N)$ is also a solution of (33).

Lemma 1 [16]**.** *The function S' defined in (26) satisfies*

$$\frac{\partial S'}{\partial Y'} - \left(\frac{\partial S'}{\partial X'}\right)^2 - 2\tilde{v}_0\frac{\partial S'}{\partial X'} = 0, \tag{34}$$

$$\frac{\partial S'}{\partial T'} - \left(\frac{\partial S'}{\partial X'}\right)^3 - 3\tilde{v}_0\left(\frac{\partial S'}{\partial X'}\right)^2 - 3\left[\tilde{v}_0^2 + \tilde{v}_{-1}\right]\frac{\partial S'}{\partial X'} = 0. \tag{35}$$

Here we identify T_2', T_3' as Y' and T'.

Proof. First of all, we note that $\partial S'/\partial T_i'$ are solutions of (21) from (30) and the ring of symmetry property. Furthermore, (26)–(29) imply

$$\frac{\partial S'}{\partial X'} = \frac{1}{\lambda} + \frac{\partial\hat{S}_0'}{\partial X'} + \lambda\frac{\partial\hat{S}_1'}{\partial X'} + \cdots, \tag{36}$$

$$\frac{\partial S'}{\partial Y'} = \frac{1}{\lambda^2} + \frac{\partial\hat{S}_0'}{\partial Y'} + \lambda\frac{\partial\hat{S}_1'}{\partial Y'} + \cdots, \tag{37}$$

$$\frac{\partial S'}{\partial T'} = \frac{1}{\lambda^3} + \frac{\partial\hat{S}_0'}{\partial T'} + \lambda\frac{\partial\hat{S}_1'}{\partial T'} + \cdots, \tag{38}$$

as $\lambda \to 0$, and

$$\frac{\partial S'}{\partial T_j'} \to 0, \qquad \text{as} \quad \lambda \to \infty. \tag{39}$$

Then (34), (35) can be directly derived from (36)–(39) and the Vekua theorem with

$$\tilde{v}_0 = -\frac{\partial\hat{S}_0'}{\partial X'}, \qquad \tilde{v}_{-1} = -\frac{\partial\hat{S}_1'}{\partial X'}.$$

□

Lemma 2. *There exist constants A, B and C such that*

$$\frac{\partial S}{\partial Y} + A\left(\frac{\partial S}{\partial X}\right)^2 = 0, \tag{40}$$

$$\frac{\partial S}{\partial T} + B\left(\frac{\partial S}{\partial X}\right)^3 + C\left(\frac{\partial S}{\partial X}\right)^2 = 0. \tag{41}$$

Here we identify T_2, T_3 as Y and T.

Proof. Using (20), (14), (17), (18), we have

$$\frac{\partial S}{\partial X} = \frac{1}{\lambda}\frac{\partial \tilde{f}}{\partial X} + \frac{\partial \hat{S}_0}{\partial X} + \lambda \frac{\partial \hat{S}_1}{\partial X} + \cdots, \tag{42}$$

$$\frac{\partial S}{\partial Y} = \frac{1}{\lambda^2} + \frac{1}{\lambda}\frac{\partial \tilde{f}}{\partial Y} + \frac{\partial \hat{S}_0}{\partial Y} + \lambda \frac{\partial \hat{S}_1}{\partial Y} + \cdots, \tag{43}$$

$$\frac{\partial S}{\partial T} = \frac{1}{\lambda^3} + \frac{1}{\lambda}\frac{\partial \tilde{f}}{\partial T} + \frac{\partial \hat{S}_0}{\partial T} + \lambda \frac{\partial \hat{S}_1}{\partial T} + \cdots, \tag{44}$$

as $\lambda \to 0$. Also from (17), we have the properties

$$\frac{\partial S}{\partial T_j} \to 0, \qquad \text{as} \quad \lambda \to \infty. \tag{45}$$

Then using (33), (42)–(44), (45), Vekua's theorem and the ring of symmetry property, we can find A, B, C, r_1 and r_2 to assure

$$\frac{\partial S}{\partial Y} + A\left(\frac{\partial S}{\partial X}\right)^2 + r_1 \frac{\partial S}{\partial X} = 0, \qquad \frac{\partial S}{\partial T} + B\left(\frac{\partial S}{\partial X}\right)^3 + C\left(\frac{\partial S}{\partial X}\right)^2 + r_2 \frac{\partial S}{\partial X} = 0.$$

To show $r_1 = r_2 = 0$, we plug (25) and the reciprocal formula (23) in the above formula. This leads to

$$\frac{\partial S'}{\partial Y'} - \frac{\Phi'_{Y'}}{\Phi'_{X'}}\frac{\partial S'}{\partial X'} + A\left(\frac{1}{\Phi'_{X'}}\frac{\partial S'}{\partial X'}\right)^2 + r_1\left(\frac{1}{\Phi'_{X'}}\frac{\partial S'}{\partial X'}\right) = 0, \tag{46}$$

$$\frac{\partial S'}{\partial T'} - \frac{\Phi'_{T'}}{\Phi'_{X'}}\frac{\partial S'}{\partial X'} + B\left(\frac{1}{\Phi'_{X'}}\frac{\partial S'}{\partial X'}\right)^3 + C\left(\frac{1}{\Phi'_{X'}}\frac{\partial S'}{\partial X'}\right)^2 + r_2\left(\frac{1}{\Phi'_{X'}}\frac{\partial S'}{\partial X'}\right) = 0. \tag{47}$$

Now using (24), we compute

$$\frac{\Phi'_{Y'}}{\Phi'_{X'}} = 2\tilde{v}_0, \qquad \frac{\Phi'_{T'}}{\Phi'_{X'}} = 3\left(\tilde{v}_0^2 + \tilde{v}_{-1}\right).$$

Plugging the above formula into (46), (47) and using lemma 1, we prove $r_1 = r_2 = 0$. □

Proposition 2. *The dHD equation*

$$\tilde{u}_{1T} = \frac{3}{4}\tilde{u}_1^{-1}\left[\tilde{u}_1^2\partial_X^{-1}\left(\frac{\tilde{u}_{1Y}}{\tilde{u}_1^2}\right)\right]_Y, \tag{48}$$

which is just the quasiclassical limit of the ordinary $(2+1)$-dimensional HD equation (1) by dropping the dispersion term, can be derived from the quasiclassical $\bar{\partial}$-problem (21).

Proof. Equating the $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$ coefficients of (40) and using (42)–(44), we get

$$A = -\frac{1}{\tilde{f}_X^2}, \qquad \frac{\partial \tilde{f}_X}{\partial Y} = \left(\frac{2}{\tilde{f}_X}\frac{\partial \hat{S}_0}{\partial X}\right)_X. \tag{49}$$

Now we define

$$\tilde{u}_1 = \frac{1}{\tilde{f}_X}, \qquad u_0 = -\tilde{u}_1\frac{\partial \hat{S}_0}{\partial X}, \qquad \tilde{u}_{-1} = -\frac{\partial \hat{S}_1}{\partial X}, \tag{50}$$

or

$$f = \int^X \frac{1}{\tilde{u}_1}, \qquad \hat{S}_0 = -\int^X \frac{\tilde{u}_0}{\tilde{u}_1}, \qquad \hat{S}_1 = -\int^X \tilde{u}_{-1}.$$

Hence (49) implies

$$\frac{\partial \tilde{u}_1}{\partial Y} = 2\tilde{u}_1^2 \tilde{u}_{0X}. \tag{51}$$

Furthermore, equating the λ^0 coefficient of (40) and using (42)–(50), we obtain

$$\int^X \left(\frac{\tilde{u}_0}{\tilde{u}_1}\right)_Y = -\tilde{u}_1^2 \left(\frac{\tilde{u}_0^2}{\tilde{u}_1^2} + \frac{2}{\tilde{u}_1}\hat{S}_{1X}\right),$$

which, by (51), is equivalent to

$$\frac{\tilde{u}_{0Y}}{\tilde{u}_1} = 2(-\tilde{u}_1 \hat{S}_{1X})_X.$$

Combining (50) with the above equation, we have

$$\frac{\partial \tilde{u}_0}{\partial Y} = 2\tilde{u}_1(\tilde{u}_1\tilde{u}_{-1})_X. \tag{52}$$

Similarly, the $\frac{1}{\lambda^3}$ and $\frac{1}{\lambda^2}$-coefficients of (41) yield

$$B = -\frac{1}{\tilde{f}_X^3}, \qquad C = -3\tilde{u}_1^2\tilde{u}_0.$$

Therefore the $\frac{1}{\lambda}$-coefficient of (41) and (50)–(52) imply

$$\frac{\partial \tilde{f}}{\partial T} = -3\left(\tilde{u}_0^2 + \tilde{u}_1\tilde{u}_{-1}\right),$$

or equivalently

$$\frac{\partial \tilde{u}_1}{\partial T} = 3\tilde{u}_1^2\left(\tilde{u}_0^2 + \tilde{u}_1\tilde{u}_{-1}\right)_X. \tag{53}$$

Finally, using (51), (52) to eliminate $\tilde{u}_0$ and $\tilde{u}_{-1}$ in (53), one obtains the dHD equation (48). □

Similarly, we can use the analytical properties $\partial S/\partial T_n = 1/\lambda^n + \tilde{f}_{T_n}/\lambda + \partial\hat{S}_0/\partial T_n + (\partial\hat{S}_1/\partial T_n)\lambda + \cdots$ and the Vekua theorem to get

$$\frac{\partial S}{\partial T_n} - \frac{1}{\tilde{f}_X^n}\left(\frac{\partial S}{\partial X}\right)^n - \sum_{k=2}^{n-1} V_{nk}(\vec{T})\left(\frac{\partial S}{\partial X}\right)^k = 0, \qquad n = 2, 3, \ldots. \tag{54}$$

Equating the coefficients of (54) as $\lambda \to 0$ and computing their compatibility condition then gives us higher flows in the dHD hierarchy. On the other hand, if we set $p = \partial S/\partial X$, then (54) is transformed into

$$\begin{aligned} &\frac{\partial p}{\partial T_n} = \frac{\partial B_n}{\partial X}, \qquad n = 1, 2, \ldots, \\ &B_n = (\mathcal{L}^n)_{\geqslant 2}, \qquad \mathcal{L} = \sum_{k\leqslant 1} \tilde{u}_k p^k. \end{aligned} \tag{55}$$

Moreover (55) is equivalent to

$$\frac{\partial \mathcal{L}}{\partial T_n} = \{B_n, \mathcal{L}\}, \qquad n = 1, 2, \ldots, \tag{56}$$

by noting that they have the same compatibility condition. Here we write the first two dHD flows (t^2-, t^3-flows) by equating the compatibility condition of (56) for $n = 2, 3$. That is:

$$\frac{\partial \tilde{u}_{-k+1}}{\partial Y} = 2\tilde{u}_1^2\frac{\partial \tilde{u}_{-k}}{\partial X} + 2k\tilde{u}_1\tilde{u}_{-k}\frac{\partial \tilde{u}_1}{\partial X}, \qquad k \geqslant 0, \tag{57}$$

$$\frac{\partial \tilde{u}_{-k+1}}{\partial T} = 3\tilde{u}_1^3\frac{\partial \tilde{u}_{-k-1}}{\partial X} + 6\tilde{u}_1^2\tilde{u}_0\frac{\partial \tilde{u}_{-k}}{\partial X} + 3(k+1)\tilde{u}_1^2\tilde{u}_{-k-1}\frac{\partial \tilde{u}_1}{\partial X} + 3k\tilde{u}_{-k}\frac{\partial\left(\tilde{u}_1^2\tilde{u}_0\right)}{\partial X}, \qquad k \geqslant 0. \tag{58}$$

6. Explicit solutions

We will consider the quasiclassical $\bar{\partial}$-problem (21) of the special form

$$S_{\bar{\lambda}} = -\frac{\lambda^{2M}}{\bar{\lambda}^2}\theta(|\lambda| - 1)(S_\lambda)^M, \tag{59}$$

with the initial condition $S|_{\frac{1}{\lambda}=0} = \frac{a_N}{\lambda^N} + \frac{a_{N-1}}{\lambda^{N-1}} + \cdots + a_0$. Here θ is the Heaviside function.

Under the transformation $\lambda \to \frac{1}{z}$, the above quasiclassical $\bar{\partial}$-problem is transformed into

$$S_{\bar{z}} = \theta(1 - |z|)(S_z)^M, \tag{60}$$

and $S|_{\bar{z}=0} = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_0$. Therefore the discussion of the previous section implies that the dHD flows (54) are derived from the analytical properties:

$$S \to \mathcal{O}(|z|), \qquad \text{as} \quad z \to 0, \tag{61}$$

$$S \to T_N z^N + \cdots + T_2 z^2 + \tilde{f}(\vec{T})z + \hat{S}_0 + \frac{\hat{S}_1}{z} + \cdots, \qquad \text{as} \quad z \to \infty. \tag{62}$$

Hence we can follow the method of characteristic used in [16] to solve (60) and get explicit solutions of the dHD hierarchy.

Example 1. The case $(M, N) = (2, 2)$: applying the method of characteristic, we obtain

$$S = \begin{cases} \frac{1}{2}\frac{(z-b)^2}{a-2\bar{z}} - c, & |z| \leqslant 1 \\ \frac{1}{2}\frac{z(z-b)^2}{az-2} - c, & |z| \geqslant 1. \end{cases} \tag{63}$$

Plugging condition (61) into the first equation of (63), we get $c = \frac{b^2}{2a}$. So the second equation of (63) implies

$$S = \frac{1}{2a}z^2 + \left(\frac{1}{a^2} - \frac{b}{a}\right)z + \frac{2}{a^3} - \frac{2b}{a^2} + \mathcal{O}\left(\frac{1}{z}\right), \qquad \text{as} \quad z \to \infty.$$

Comparing the above formula with (62) and identifying $Y = T_2$, we then have

$$a = \frac{1}{2Y}, \qquad b = 2Y - \frac{\tilde{f}}{2Y}, \qquad c = Y\left(2Y - \frac{\tilde{f}}{2Y}\right)^2, \qquad \hat{S}_0 = 4Y\tilde{f}. \tag{64}$$

To solve $\tilde{f}$, we use the dynamic equation of the first flow (deriving from (61), (62) as (49) in the proof of proposition 2):

$$\tilde{f}_Y = 2\frac{\hat{S}_{0X}}{\tilde{f}_X}. \tag{65}$$

Plugging (64) into (65), we obtain

$$\tilde{f} = 4Y^2 + g(X), \qquad b = -\frac{g}{2Y}, \qquad c = \frac{g^2}{4Y}, \qquad \hat{S}_0 = 4Y(4Y^2 + g), \tag{66}$$

where g is an arbitrary function on X. Now differentiating both sides of the second equation of (63) with respect to X, using (64), (66) and recalling the definition of $p = \partial S/\partial X$, we then derive

$$z = \frac{1}{2}\left(b - \frac{a(p + c_X)}{b_X} + \sqrt{\left[b - \frac{a(p + c_X)}{b_X}\right]^2 + \frac{8}{b_X}(p + c_X)}\right)$$
$$= \frac{p}{2g_X} + \frac{1}{2}\sqrt{\frac{p^2}{g_X^2} - \frac{16Yp}{g_X} - 8g}$$

$$= \frac{p}{2g_X} + \frac{1}{2}\left(\frac{p}{g_X} - 8Y\right)\left(1 - \frac{64Y^2 + 8g}{\left(\frac{p}{g_X} - 8Y\right)^2}\right)^{\frac{1}{2}}$$

$$= \frac{p}{g_X} - 4Y - (16Y^2 + 2g)\frac{g_X}{p}\left(1 + \frac{8Yg_X}{p} + \left(\frac{8Yg_X}{p}\right)^2 + \cdots\right)$$

$$- (16Y^2 + 2g)^2\frac{g_X^3}{p^3}\left(1 + \frac{8Yg_X}{p} + \left(\frac{8Yg_X}{p}\right)^2 + \cdots\right)^3 + \cdots = \sum_{k\leqslant 1}\tilde{u}_k p^k$$

from which we have

$$\begin{aligned} &\tilde{u}_1 = \frac{1}{g_X}, \qquad \tilde{u}_0 = -4Y, \qquad \tilde{u}_{-1} = -g_X(16Y^2 + 2g), \\ &\tilde{u}_{-2} = -8Yg_X^2(16Y^2 + 2g), \\ &\tilde{u}_{-3} = -g_X^3(16Y^2 + 2g)(80Y^2 + 2g), \ldots. \end{aligned} \tag{67}$$

Here g is an arbitrary function of X. One can check that (67) satisfies (57). Hence we obtain an explicit solution for the first dHD flow.

Finally, we note

$$z = \frac{\tilde{u}_1}{2}p + \frac{1}{2}\sqrt{\tilde{u}_1^2 p^2 + 4\tilde{u}_1\tilde{u}_0 p - 8g},$$

which is a two-reduction of the dHD hierarchy. This reduction cannot be obtained by standard hodograph methods [11, 13] since

$$\frac{\partial z(p,\tilde{u})}{\partial p} = \frac{\tilde{u}_1\left(\tilde{u}_1 p + 2\tilde{u}_0 + \sqrt{(\tilde{u}_1 p + 2\tilde{u}_0)^2 - 4\tilde{u}_0^2 - 8g}\right)}{2\sqrt{(\tilde{u}_1 p + 2\tilde{u}_0)^2 - 4\tilde{u}_0^2 - 8g}}$$

which has no zeros provided that $\tilde{u}_1 \neq 0$ and hence the two-reduction HD system cannot be diagonalized through the Riemann invariants $z(p_i)$ where p_i are zeros of $\partial z/\partial p = 0$.

Example 2. The case $(M, N) = (3, 2)$: let $S_z|_{\bar{z}=0} = \frac{z}{a} + b$ and $Y = T_2$. Using the method of characteristic and condition (61), we obtain

$$\begin{aligned} S &= -\frac{2m^3}{z} + \frac{a}{2}m^2 - \frac{ab^2}{2}, \qquad |z| < 1, \\ m &= \frac{1}{6\bar{z}}\left(a - \sqrt{a^2 - 12(z + ab)\bar{z}}\right). \end{aligned}$$

Holomorphically extending to $|z| > 1$ and comparing with the expansion

$$S = Yz^2 + \tilde{f}z + \hat{S}_0 + \hat{S}_1/z + \cdots$$

as $z \to \infty$, we get

$$\begin{aligned} 108Y &= -a(a^2 - 18) + (a^2 - 12)^{3/2}, \\ \tilde{f} &= \frac{ab}{6}\left(a - \sqrt{a^2 - 12}\right), \\ \hat{S}_0 &= -\frac{ab^2}{2}\left(1 - \frac{a}{\sqrt{a^2 - 12}}\right), \\ \hat{S}_1 &= a^3b^3(a^2 - 12)^{-3/2}. \end{aligned}$$

Hence

$$a = \frac{1}{6Y}(D^{1/3} + (216Y^2 + 1)D^{-1/3} + 1),$$
$$D := -5832Y^4 - 540Y^2 + 1 + 24\sqrt{3}Y\sqrt{19\,683Y^6 - 2187Y^4 + 81Y^2 - 1},$$
$$b = \frac{6\tilde{f}}{a(a - \sqrt{a^2 - 12})}.$$

It turns out that

$$\tilde{f}_Y = \frac{2}{\tilde{f}_X}\frac{\partial \hat{S}_0}{\partial X} = \frac{72\tilde{f}}{a\sqrt{a^2 - 12}(a - \sqrt{a^2 - 12})}$$

which implies

$$\tilde{f}(X, Y) = g(X)\exp(\omega(Y)),$$

where $g(X)$ is an arbitrary function in X and

$$\omega(Y) = \int^Y \mathrm{d}Y' \frac{72}{a\sqrt{a^2 - 12}(a - \sqrt{a^2 - 12})}.$$

Finally the primary variable $\tilde{u}_1$ is given by

$$\tilde{u}_1 = \frac{1}{\tilde{f}_X} = \frac{\exp(-\omega(Y))}{g_X}.$$

Furthermore,

$$\tilde{u}_0 = -\tilde{u}_1\hat{S}_{0X} = \frac{-36g\,\mathrm{e}^{\omega(Y)}}{a\sqrt{a^2 - 12}(a - \sqrt{a^2 - 12})},$$
$$\tilde{u}_{-1} = -\hat{S}_{1X} = \frac{-648g^2 g_X\,\mathrm{e}^{3\omega(Y)}}{(a^2 - 12)^{3/2}(a - \sqrt{a^2 - 12})^3},$$
$$\cdots$$

It can be verified that $\tilde{u}_1, \tilde{u}_0, \tilde{u}_{-1}$ satisfy (57). And we obtain another explicit solution of the first dHD flow which is not degenerate (as that in example 1).

Example 3. The case $(M, N) = (2, 3)$: let $S_z|_{\bar{z}=0} = az^2 + bz + c$ and $Y = T_2, T = T_3$. Similarly, using the method of characteristic and condition (61), we obtain the expansion as $z \to \infty$:

$$S = Tz^3 + Yz^2 + \tilde{f}(X, Y, T)z + \hat{S}_0(X, Y, T) + \hat{S}_1(X, Y, T)/z + \cdots$$

with

$$3T = -\frac{3}{4} + \frac{3}{8a} + \frac{1}{32a^2}((1 - 8a)^{3/2} - 1),$$
$$2Y = \frac{b}{8a^2}(1 - 4a - \sqrt{1 - 8a}),$$
$$\tilde{f}(X, Y, T) = \frac{c}{4a}(1 - \sqrt{1 - 8a}) - \frac{b^2}{8a^2}\left(1 + \frac{4a - 1}{\sqrt{1 - 8a}}\right),$$
$$\hat{S}_0 = -\frac{bc}{2a}\left(1 - \frac{1}{\sqrt{1 - 8a}}\right) + \frac{b^3}{12a^2}\left(1 - \frac{1 - 12a}{(1 - 8a)^{3/2}}\right),$$
$$\hat{S}_1 = \frac{1}{\sqrt{1 - 8a}}\left(\frac{2b^2}{1 - 8a} + c\right)^2.$$

So

$$a(T) = \frac{(36T+1) \pm (1-12T)^{3/2}}{18(1+4T)^2},$$
$$b(Y,T) = \frac{16a^2 Y}{1-4a-\sqrt{1-8a}},$$
$$c(X,Y,T) = \frac{4a}{1-\sqrt{1-8a}}\left(\tilde{f}(X,Y,T) - \frac{32a^2Y^2}{\sqrt{1-8a}(1-4a-\sqrt{1-8a})}\right).$$

To determine $\tilde{f}$, we plug the above expression into the dynamic equation (deriving from (40), (41) as in the proof of lemma 1):

$$\tilde{f}_Y = \frac{2\hat{S}_{0X}}{\tilde{f}_X}, \qquad \tilde{f}_T = 3\frac{S_{1X}}{\tilde{f}_X} - 3\frac{S_{0X}^2}{\tilde{f}_X^2}, \tag{68}$$

and obtain

$$\tilde{f}_Y = \frac{4b}{\sqrt{1-8a}} = \frac{64a^2}{\sqrt{1-8a}(1-4a-\sqrt{1-8a})}Y.$$

So

$$\tilde{f}(X,Y,T) = \frac{32a^2}{\sqrt{1-8a}(1-4a-\sqrt{1-8a})}Y^2 + g(X,T).$$

Hence

$$\tilde{f}_T = g_T + \left(\frac{32a^2}{\sqrt{1-8a}(1-4a-\sqrt{1-8a})}\right)_T Y^2$$
$$= \frac{48a^2}{\sqrt{1-8a}(1-4a-\sqrt{1-8a})}g + \mathcal{O}(Y^2)$$
$$= \frac{6\sqrt{1-8a}}{1-12T}g + \mathcal{O}(Y^2)$$

which implies

$$g(X,T) = C(X)\,\mathrm{e}^{\omega(T)}, \qquad \omega(T) = \int^T \mathrm{d}T' \frac{6\sqrt{1-8a(T')}}{1-12T'},$$

for an arbitrary function $C(X)$. Finally the primary variable $\tilde{u}_1$ is given by

$$\tilde{u}_1 = \frac{1}{\tilde{f}_X} = \frac{\mathrm{e}^{-\omega(T)}}{C'(X)}.$$

Hence

$$\tilde{u}_0 = -\frac{64a^2Y}{\sqrt{1-8a}(1-\sqrt{1-8a})^2},$$
$$\tilde{u}_{-1} = \frac{8a}{1-8a-\sqrt{1-8a}}\left(\frac{2b^2}{1-8a} + \frac{4a}{1-\sqrt{1-8a}}C(X)\,\mathrm{e}^{\omega(T)}\right)\mathrm{e}^{\omega(T)}C'(X),$$
$$\ldots.$$

We can justify that $\tilde{u}_1, \tilde{u}_0, \tilde{u}_{-1}$ satisfy (57), (58) for $k = 0$.

To conclude this section, we remark that for any $(M, N, s) = (M, 3, s)$, $M > 0$, $s \geqslant 0$, it is impossible to get any (2 + 1)-dHD equation from the Beltrami equation

$$S_{\bar{z}} = \theta(1-|z|)\bar{z}^s \sum_{m\geqslant 0}^{M} p_m(z)(S_z)^m, \tag{69}$$

with

$$S = \sum_{n\geqslant 0} c_n(z)\bar{z}^{n(s+1)}, \qquad |z| < 1, \tag{70}$$

$$S|_{\bar{z}=0} = c_0 = a_1 z + a_2 z^2 + a_3 z^3. \tag{71}$$

First of all, substituting (70) into (69), we obtain the recursive formula

$$c_{n+1} = \frac{1}{(n+1)(s+1)} \sum_{m\geqslant 0}^{M} p_m(z) \left(\sum_{r_1+\cdots+r_m=n} c'_{r_1}\cdots c'_{r_m} \right), \qquad n \geqslant 0.$$

Also, to have the expansion,

$$S = Tz^3 + Yz^2 + \tilde{f}(X,Y,T)z + \hat{S}_0(X,Y,T) + \hat{S}_1(X,Y,T)/z + \cdots,$$

as $z \to \infty$, we impose condition [18]

$$\deg p_M = 0, \quad \deg p_{M-1} = 2, \quad \deg p_{M-2} = 4, \ldots. \tag{72}$$

Thus the recursive formula and (71) imply

$$\begin{aligned}
c_1 &= \frac{1}{s+1}\left\{ p_M \underbrace{c'_0 \times \cdots \times c'_0}_{M} + p_{M-1}\underbrace{c'_0 \times \cdots \times c'_0}_{M-1} + \cdots + p_1 c'_0 + p_0 \right\} \\
&= \frac{1}{s+1}\{p_M[(3a_3)^M z^{2M} + (3a_3)^{M-1}(2a_2)z^{2M-1} \\
&\quad + \{(3a_3)^{M-1}a_1 + (3a_3)^{M-2}(2a_2)^2\}z^{2M-2} \\
&\quad + \{(3a_3)^{M-2}(2a_2)a_1 + (3a_3)^{M-3}(2a_2)^3\}z^{2M-3} + \cdots] \\
&\quad + p_{M-1}[(3a_3)^{M-1}z^{2(M-2)} + \cdots] \\
&\quad \cdots \\
&\quad + p_1(3a_3z^2 + 2a_2z + a_1) + p_0\} \\
&= P_{2M}(a_3)z^{2M} + P_{2M-1}(a_3,a_2)z^{2M-1} \\
&\quad + (P_{2M-2,1}(a_3)a_1 + P_{2M-2,2}(a_3,a_2))z^{2M-2} \\
&\quad + (P_{2M-3,1}(a_3,a_2)a_1 + P_{2M-3,2}(a_3,a_2))z^{2M-3} \\
&\quad + \text{lower order terms in } z,
\end{aligned}$$

where P_k, $P_{k,i}$ are polynomials. Similarly, the coefficients of the four leading z-terms in c_n are respectively of the form

$$Q_3(a_3), \qquad Q_2(a_3,a_2),$$
$$Q_{1,1}(a_3,a_2) + Q_{1,2}(a_3,a_2)a_1, \qquad Q_{0,1}(a_3,a_2) + Q_{0,2}(a_3,a_2)a_1.$$

Therefore, plugging these c_n into (70) and using (72), for $\forall M, \forall s$, as $z \to \infty$, the four leading z-terms of S are

$$R_3(a_3)z^3 = Tz^3, \qquad R_2(a_3,a_2)z^2 = Yz^2,$$
$$(R_{11}(a_3)a_1 + R_{12}(a_3,a_2))z = \tilde{f}z, \qquad R_{01}(a_3,a_2)a_1 + R_{12}(a_3,a_2) = \hat{S}_0,$$

where $R_i R_{kj}$ are the polynomials. So

$$a_3 = a_3(T), \qquad a_2 = a_2(Y,T), \qquad a_1 = a_1(X,Y,T),$$

and hence the dynamic equation (68) yields

$$\tilde{f}_Y = 2\frac{\hat{S}_{0X}}{\tilde{f}_X} = 2\frac{R_{01}a_{1X}}{R_{11}a_{1X}} = 2\frac{R_{01}}{R_{11}}(Y, T).$$

Integrating both sides, we then derive $\tilde{f}(X, Y, T) = F(Y, T) + G(X, T)$. So the dHD solution obtained is $\tilde{u}_1 = 1/\tilde{f}_X = 1/G_X$ which depends only on (X, T). Comparison of these solutions with the hodograph method in [3, 22] could be interesting and needs further investigation.

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